Calc BC Parametric WS1

Name: Block: Seat:

Parametric Functions Parametric functions are another way of viewing functions. This time, the values of x and y are both dependent on another independent variable, usually t, for time. With parametric functions, we can draw very interesting curves such as the **asteroid**.

Cartesian Equation	Parametric Equation
$x^{2/3} + y^{2/3} = 4^{2/3}$	$x(t) = 4\cos^3 t$ $y(t) = 4\sin^3 t$

- Parametric equations can be used to solve physical problems, and thus have many interesting real world applications. We like thinking about movement in two dimensions with respect to time, which is done often with parametrics.
- Parametric equations have obvious uses in real life. While many concepts that we learned are similar in this regard, parametric equations offer a physical representation that few concepts do. Visualizing the motion of objects of on a plane is easy and models many things in the real world.
- Parametric equations allow in depth analysis of projectile motion and provides the way one can derive kinematics equations.
- We can use calculus to determine the slope of the curve at any point, to write an equation line tangent to the curve at a given point, and to find the length of a curve between two points.

The following formulas will be used on the AP Calculus exam. You need to know them.

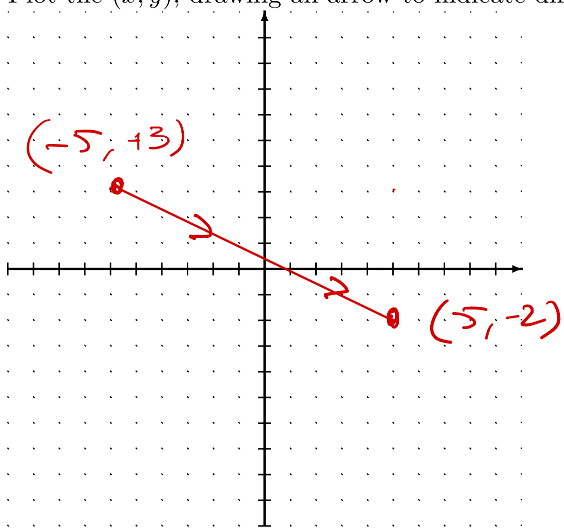
The first derivative (the change in y with respect to x)	$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$
The second derivative of y with respect to x	$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$ ✓ ✓
The speed of a particle at a given value of t	$\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$
The arc length	$L = \displaystyle\int_{t_1}^{t_2} \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$

1. $x(t) = 2t - 1$
   $y(t) = 1 - t$
   $t \in [-2, 3]$

   (a) Make a table with columns for x, y, and t

t	-2	-1	0	1	2	3
x	-5	-3	-1	1	3	5
y	3	2	1	0	-1	-2

   (b) Plot the (x, y), drawing an arrow to indicate direction



$x = 2t - 1$
$y = 1 - t$

   (c) Eliminate the parameter

$x = 2t - 1 \qquad y = 1 - t$

$t = 1 - y$

$x = 2(1 - y) - 1 \longrightarrow \frac{x+1}{2} = 1 - y$

$y = 1 - \frac{x+1}{2}$

   (d) Find the highest point of the curve

$\frac{d}{dt}(1 - t) = -1$

no C.P.

t	-2	3
$y(t)$		

   (e) Find the length of the curve on the interval

(pyth) $= \sqrt{(10)^2 + (5)^2} = \sqrt{125} = \boxed{5\sqrt{5}}$

Calculus $= \int_{-2}^{3} \sqrt{(2)^2 + (-1)^2}\, dt = \int_{-2}^{3} \sqrt{5}\, dt$

$= \sqrt{5} \int_{-2}^{3} dt$

$\sqrt{5}(t)\Big|_{-2}^{3} = \sqrt{5}(3 + 2) = 5\sqrt{5}$

$x'(t) = 2t \qquad y'(t) = \frac{1}{2}$

2. $x(t) = t^2 - 4$
   $y(t) = \dfrac{t}{2}$
   $t \in [-2, 4]$

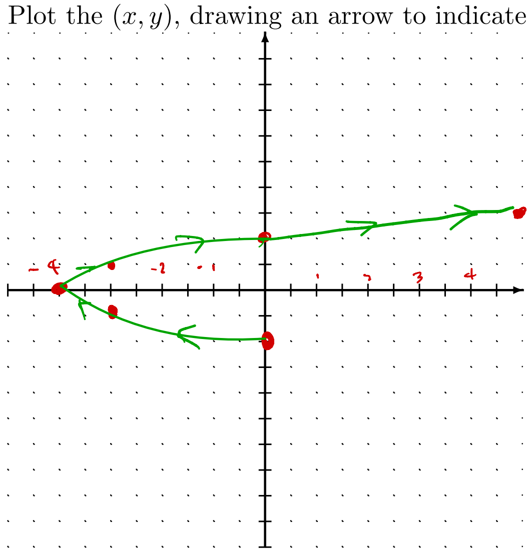
$t = 2y$

$x = (2y)^2 - 4$

(a) Make a table with columns for x, y, and t

t	-2	-1	0	1	2	3	4
x	0	-3	-4	-3	0	5	12
y	-1	-1/2	0	1/2	1	3/2	2

(b) Plot the (x, y), drawing an arrow to indicate direction



(c) Eliminate the parameter

(d) Find the left most point of the curve

Cut Test

t	
$x(t)$	

(e) Find the length of the curve on the interval

$$\int_{-2}^{4} \sqrt{(2t)^2 + \left(\tfrac{1}{2}\right)^2}\, dt$$

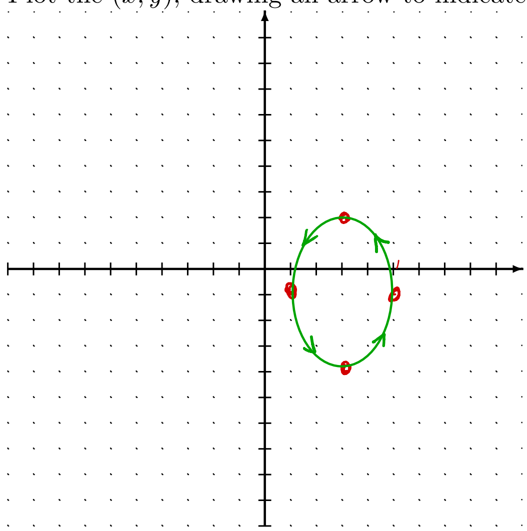
$$\left(\frac{y+1}{3}\right)^2 + \left(\frac{x-3}{2}\right)^2 = 1$$

3. $x(t) = 3 + 2\cos t$ $\longrightarrow$ $\frac{x-3}{2} = \cos t$
   $y(t) = -1 + 3\sin t$ $\longrightarrow$ $\frac{y+1}{3} = \sin t$
   $t \in [0, 2\pi]$

   (a) Make a table with columns for x, y, and t

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	5	3	1	3	5
y	-1	2	-1	-4	-1

   (b) Plot the (x, y), drawing an arrow to indicate direction



   (c) Eliminate the parameter (slove for $\sin t$ and $\cos t$ then use the identity $\cos^2 t + \sin^2 t = 1$)

   (d) Find the lowest point of the curve

$y' = 3\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$ are CP

t	0	$\pi/2$	$3\pi/2$
$y(t)$	-1	2	-4

2π

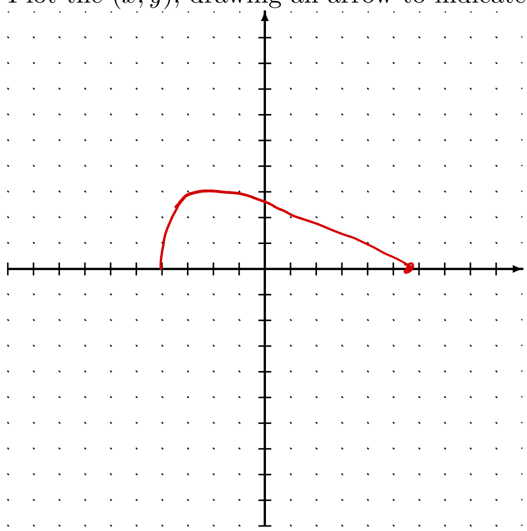
   (e) Find the length of the curve on the interval

4. $x(t) = t^2 - 4$
   $y(t) = 3\sin t$
   $t \in [0, \pi]$

   (a) Make a table with columns for x, y, and t

t	0	$\frac{\pi}{2}$	π
x	-4	-1.5	5.870
y	0	3	0

   (b) Plot the (x, y), drawing an arrow to indicate direction



   (c) Find the lowest point of the curve

Cand Test

$\frac{dy}{dt} = 3\cos t$, CP: $\frac{\pi}{2}$

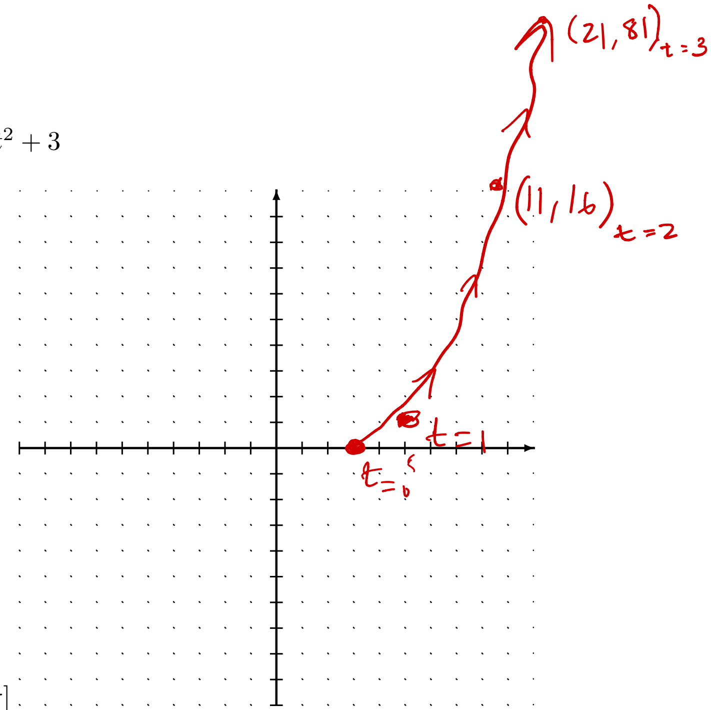
t	0	$\frac{\pi}{2}$	π
$y(t)$	0	3	0

min @ $(4, 0)$, $\left(\frac{\pi^2}{4} - 4, 0\right)$

   (d) Find the length of the curve on the interval

$$\int_0^{\pi} \sqrt{(2t)^2 + (3\cos t)^2}\, dt$$

5. $x(t) = 2t^2 + 3$
   $y(t) = t^4$



$t \in [0, 2\pi]$

(a) Write the equation for the line tangent to the path of the function when $t = -1$. (The tangent line will be in terms of x and y only.)

$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{4t} = t^2 \Big|_{t=-1} = 1$

$x(-1) = 2(-1)^2 + 3 = 5$

$y(-1) = (-1)^4 = 1$

$y - 1 = 1(x - 5)$

$y = x - 4$

6. Given the parametric function $x(t) = \frac{1}{t}$
   $y(t) = -2 + \ln t$

(a) Find $\frac{dy}{dx}$ in terms of t

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t}}{\frac{-1}{t^2}} = \frac{1}{t} \cdot \frac{t^2}{-1} = -t$

(b) Find $\frac{d^2y}{dx^2}$ in terms of t

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{-1}{-\frac{1}{t^2}} = t^2$

7. (From 1998 BC6 (calculator allowed)) A particle moves along the curve defined by the equation $y = x^3 - 3x$ The x-coordinate of the particle, $x(t)$ satisfies the equation $\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$ for $t \geq 0$ with the initial condition $x(0) = -4$

points

3

$u = 2t+1$ $\quad u(0) = 1$

$\frac{1}{2}\,du = dx$ $\quad u(t) = 2t+1$

(a) Find $x(t)$ in terms of t

$$x(t) = -4 + \int_0^t \frac{dx}{dt}\,dt = -4 + \frac{1}{2}\int_1^{2t+1} u^{-1/2}\,du$$

$$= -4 + \sqrt{u}\,\Big|_1^{2t+1}$$

$$= -4 + \sqrt{2t+1} - 1 = -5 + \sqrt{2t+1}$$

2

(b) Find $\frac{dy}{dt}$ in terms of t

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt} - 3\frac{dx}{dt}$$

$$= (3x^2 - 3)\frac{dx}{dt}$$

$$= \left(3\left(-5+\sqrt{2t+1}\right)^2 - 3\right)\left(\frac{1}{\sqrt{2t+1}}\right)$$

4

(c) Find the location and speed of the particle at time $t = 4$

$$x(4) = \sqrt{9} - 5 = -2$$

@ $t=4$, $x=-2$ → $y(-2) = (-2)^3 - 3(-2) = -2$

location @ $t=4$ is $(-2, -2)$

$$\left.\frac{dx}{dt}\right|_{t=4} = \frac{1}{3}$$

$$\left.\frac{dy}{dt}\right|_{t=4} = \frac{3(3-5)^2 - 3}{3} = 3$$

$$\text{Speed} = \sqrt{\left(\tfrac{1}{3}\right)^2 + 3^2} = \frac{\sqrt{82}}{3} = 3.018$$